

On exponential cosmological type solutions in the model with Gauss-Bonnet term and variation of gravitational constant

V. D. Ivashchuk ¹ and A. A. Kobtsev ²,

*Center for Gravitation and Fundamental Metrology, VNIIMS, 46 Ozyornaya ul.,
Moscow 119361, Russia*

*Institute of Gravitation and Cosmology, Peoples' Friendship University of Russia, 6
Miklukho-Maklaya ul., Moscow 117198, Russia*

Abstract

A D -dimensional gravitational model with Gauss-Bonnet term is considered. When ansatz with diagonal cosmological type metrics is adopted, we find solutions with exponential dependence of scale factors (with respect to “synchronous-like” variable) which describe an exponential expansion of “our” 3-dimensional factor-space and obey the observational constraints on the temporal variation of effective gravitational constant G . Among them there are two exact solutions in dimensions $D = 22, 28$ with constant G and also an infinite series of solutions in dimensions $D \geq 2690$ with the variation of G obeying the observational data.

¹e-mail: ivashchuk@mail.ru

²e-mail: speedyblackbird@gmail.com

1 Introduction

Here we deal with D -dimensional gravitational model with the Gauss-Bonnet term. The action reads

$$S = \int_M d^D z \sqrt{|g|} \{ \alpha_1 R[g] + \alpha_2 \mathcal{L}_2[g] \}, \quad (1.1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric defined on the manifold M , $\dim M = D$, $|g| = |\det(g_{MN})|$ and

$$\mathcal{L}_2 = R_{MNPQ} R^{MNPQ} - 4R_{MN} R^{MN} + R^2 \quad (1.2)$$

is the standard Gauss-Bonnet term. Here α_1 and α_2 are non-zero constants.

Earlier the appearance of the Gauss-Bonnet term was motivated by string theory [1, 2, 3, 4, 5]. At present, the (so-called) Einstein-Gauss-Bonnet (EGB) gravitational model and its modifications are intensively used in cosmology, see [6, 7] (for $D = 4$), [8, 9, 10, 11, 12, 13, 14, 15, 17] and references therein, e.g. for explanation of accelerating expansion of the Universe following from supernovae (type Ia) observational data [18, 19, 20]. Certain exact solutions in multidimensional EGB cosmology were obtained in [8]-[17] and some other papers.

Here we are dealing with the cosmological type solutions with diagonal metrics (of Bianchi-I-like type) governed by n scale factors depending upon one variable, where $n > 3$. Moreover, we restrict ourselves by the solutions with exponential dependence of scale factors (with respect to “synchronous-like” variable τ)

$$a_i(\tau) \sim \exp(h^i \tau), \quad (1.3)$$

$i = 1, \dots, n$; $D = n + 1$, with the aim to find solutions describing an exponential isotropic expansion of 3-dimensional flat factor-space, i.e. with

$$h^1 = h^2 = h^3 = H > 0, \quad (1.4)$$

and small enough variation of the effective gravitational constant G , which is proportional to the inverse volume scale factor of the internal space, i.e.

$$G \sim \prod_{i=4}^n [a_i(\tau)]^{-1} \sim \exp(-Int \tau), \quad (1.5)$$

where here and in what follows we denote

$$Int = \sum_{i=4}^n h^i. \quad (1.6)$$

see [21, 22, 23, 24, 25] and references therein. We call $h^i = \dot{a}_i/a_i$ as “Hubble-like” parameter corresponding to i -th subspace.

In cosmological model under consideration with anisotropic “internal space”, we get for dimensionless parameter of temporal variation of G the following relation from (1.5) and (1.6)

$$\frac{\dot{G}}{GH} = -Int/H, \quad (1.7)$$

where H is the Hubble parameter.

As for the experimental data, the variation of the gravitational constant is allowed at the level of 10^{-13} per year and less. We use the following constraint on the magnitude of the dimensionless variation of the gravitational constant

$$-0,65 \cdot 10^{-3} < \frac{\dot{G}}{GH} < 1,12 \cdot 10^{-3}, \quad (1.8)$$

which comes from the most stringent limitation on G -dot obtained by the set of ephemerides [26]

$$\dot{G}/G = (0.16 \pm 0.6) \cdot 10^{-13} \text{ year}^{-1} \quad (1.9)$$

allowed at 95% confidence ($2\text{-}\sigma$) level and the present value of the Hubble parameter [27] (which characterizes the rate of expansion of the observable Universe)

$$H_0 = (67,80 \pm 1,54) \text{ km/s Mpc}^{-1} = (6.929 \pm 0,157) \cdot 10^{-11} \text{ year}^{-1}, \quad (1.10)$$

with 95% confidence level. It should be noted that the original result for H_0 in [27] (for the Planck best-fit cosmology including external data set) was presented at 68% confidence ($1\text{-}\sigma$) level. In restriction (1.8) we use the lower allowed value for H_0 in (1.10) in order to obtain the confidence level more than 95%.

Thus, we are seeking here the cosmological solutions which obey the relations (1.3)-(1.8), listed above.

The paper is organized as follows. In Section 2 the equations of motion for D -dimensional EGB model are considered. For diagonal cosmological type metrics the equations of motion are equivalent to a set of Lagrange equations corresponding to certain “effective” Lagrangian [29, 30] (see also [9, 28]). In Section 3 some cosmological solutions with exponential behavior of scale factors satisfying the restriction (1.8) are obtained for two isotropic factor spaces and positive value of $\alpha = \alpha_2/\alpha_1$.

2 The cosmological type model and its effective Lagrangian

2.1 The set-up

Here we consider the manifold

$$M = \mathbb{R}_* \times M_1 \times \dots \times M_n, \quad (2.1)$$

with the metric

$$g = we^{2\gamma(u)} du \otimes du + \sum_{i=1}^n e^{2\beta^i(u)} \varepsilon_i dy^i \otimes dy^i, \quad (2.2)$$

where $w = \pm 1$, $\varepsilon_i = \pm 1$, $i = 1, \dots, n$, and M_1, \dots, M_n are one dimensional manifolds (either \mathbb{R} or S^1). Here and in what follows $\mathbb{R}_* = (u_-, u_+)$ is an open subset in \mathbb{R} . The functions $\gamma(u)$ and $\beta^i(u)$, $i = 1, \dots, n$, are smooth on $\mathbb{R}_* = (u_-, u_+)$.

For $w = -1$, $\varepsilon_1 = \dots = \varepsilon_n = 1$ the metric (2.2) is a cosmological one, while for $w = 1$, $\varepsilon_1 = -1$, $\varepsilon_2 = \dots = \varepsilon_n = 1$ it describes certain static configurations.

For physical applications we put $M_1 = M_2 = M_3 = \mathbb{R}$, while M_4, \dots, M_n will be considered to be compact ones (i.e. coinciding with S^1).

The integrand in (1.1), when the metric (2.2) is substituted, reads as follows

$$\sqrt{|g|} \{ \alpha_1 R[g] + \alpha_2 \mathcal{L}_2[g] \} = L + \frac{df}{du}, \quad (2.3)$$

where

$$L = \alpha_1 L_1 + \alpha_2 L_2, \quad (2.4)$$

$$L_1 = (-w) e^{-\gamma+\gamma_0} G_{ij} \dot{\beta}^i \dot{\beta}^j, \quad (2.5)$$

$$L_2 = -\frac{1}{3} e^{-3\gamma+\gamma_0} G_{ijkl} \dot{\beta}^i \dot{\beta}^j \dot{\beta}^k \dot{\beta}^l, \quad (2.6)$$

$$\gamma_0 = \sum_{i=1}^n \beta^i \quad (2.7)$$

and

$$G_{ij} = \delta_{ij} - 1, \quad (2.8)$$

$$G_{ijkl} = (\delta_{ij} - 1)(\delta_{ik} - 1)(\delta_{il} - 1)(\delta_{jk} - 1)(\delta_{jl} - 1)(\delta_{kl} - 1) \quad (2.9)$$

are respectively the components of two metrics on \mathbb{R}^n [29, 30]. The first one is the well-known “minisupermetric” - 2-metric of pseudo-Euclidean signature: $\langle v_1, v_2 \rangle = G_{ij} v_1^i v_2^j$ and the second one is the Finslerian 4-metric: $\langle v_1, v_2, v_3, v_4 \rangle = G_{ijkl} v_1^i v_2^j v_3^k v_4^l$, $v_s = (v_s^i) \in \mathbb{R}^n$, where $\langle ., . \rangle$ and $\langle ., ., ., . \rangle$ are respectively 2- and 4-linear symmetric forms on \mathbb{R}^n . Here we denote $\dot{A} = dA/du$ etc. The function $f(u)$ in (2.3) is irrelevant for our consideration (see [29, 30]).

The derivation of (2.4)-(2.6) is based on the following identities [29, 30]:

$$G_{ij} v^i v^j = \sum_{i=1}^n (v^i)^2 - \left(\sum_{i=1}^n v^i \right)^2, \quad (2.10)$$

$$\begin{aligned} G_{ijkl} v^i v^j v^k v^l &= \left(\sum_{i=1}^n v^i \right)^4 - 6 \left(\sum_{i=1}^n v^i \right)^2 \sum_{j=1}^n (v^j)^2 \\ &+ 3 \left(\sum_{i=1}^n (v^i)^2 \right)^2 + 8 \left(\sum_{i=1}^n v^i \right) \sum_{j=1}^n (v^j)^3 - 6 \sum_{i=1}^n (v^i)^4. \end{aligned} \quad (2.11)$$

It follows immediately from the definitions (2.8) and (2.9) that

$$G_{ij}v^iv^j = -2 \sum_{i<j} v^iv^j, \quad (2.12)$$

$$G_{ijkl}v^iv^jv^kv^l = 24 \sum_{i<j<k<l} v^iv^jv^kv^l. \quad (2.13)$$

2.2 The equations of motion

The equations of motion corresponding to the action (1.1) have the following form

$$\mathcal{E}_{MN} = \alpha_1 \mathcal{E}_{MN}^{(1)} + \alpha_2 \mathcal{E}_{MN}^{(2)} = 0, \quad (2.14)$$

where

$$\mathcal{E}_{MN}^{(1)} = R_{MN} - \frac{1}{2} R g_{MN}, \quad (2.15)$$

$$\begin{aligned} \mathcal{E}_{MN}^{(2)} = & 2(R_{MPQS}R_N{}^{PQS} - 2R_{MP}R_N{}^P \\ & - 2R_{MPNQ}R^{PQ} + RR_{MN}) - \frac{1}{2} \mathcal{L}_2 g_{MN}. \end{aligned} \quad (2.16)$$

It was shown in [30] that the field equations (2.14) for the metric (2.2) are equivalent to the Lagrange equations corresponding to the Lagrangian L from (2.4).

Thus, equations (2.14) read as follows

$$w\alpha_1 G_{ij}\dot{\beta}^i\dot{\beta}^j + \alpha_2 e^{-2\gamma} G_{ijkl}\dot{\beta}^i\dot{\beta}^j\dot{\beta}^k\dot{\beta}^l = 0, \quad (2.17)$$

$$\frac{d}{du} \left[-2w\alpha_1 G_{ij}e^{-\gamma+\gamma_0}\dot{\beta}^j - \frac{4}{3}\alpha_2 e^{-3\gamma+\gamma_0} G_{ijkl}\dot{\beta}^j\dot{\beta}^k\dot{\beta}^l \right] - L = 0, \quad (2.18)$$

$i = 1, \dots, n$. Due to (2.17) $L = -w\frac{2}{3}e^{-\gamma+\gamma_0}\alpha_1 G_{ij}\dot{\beta}^i\dot{\beta}^j$.

2.3 Reduction to an autonomous system of first order differential equations

Now we put $\gamma = 0$ and denote $u = \tau$, where τ is a “synchronous-like” variable. By introducing “Hubble-like” variables $h^i = \dot{\beta}^i$, the eqs. (2.17) and (2.18) may be rewritten as follows

$$w\alpha_1 G_{ij}h^ih^j + \alpha_2 G_{ijkl}h^ih^jh^kh^l = 0, \quad (2.19)$$

$$\begin{aligned} & \left[-2w\alpha_1 G_{ij}h^j - \frac{4}{3}\alpha_2 G_{ijkl}h^jh^kh^l \right] \sum_{i=1}^n h^i \\ & + \frac{d}{d\tau} \left[-2w\alpha_1 G_{ij}h^j - \frac{4}{3}\alpha_2 G_{ijkl}h^jh^kh^l \right] - L = 0, \end{aligned} \quad (2.20)$$

$i = 1, \dots, n$, where

$$L = -w\alpha_1 G_{ij} h^i h^j - \frac{1}{3}\alpha_2 G_{ijkl} h^i h^j h^k h^l. \quad (2.21)$$

Due to (2.19) $L = -\frac{2}{3}w\alpha_1 G_{ij} h^i h^j$.

Thus, we are led to the autonomous system of the first-order differential equations on $h^1(\tau), \dots, h^n(\tau)$ [30, 30].

Here and in what follows we use relations (2.10), (2.11) and the following formulas

$$G_{ij} v^j = v^i - S_1, \quad (2.22)$$

$$G_{ijkl} v^j v^k v^l = S_1^3 + 2S_3 - 3S_1 S_2 + 3(S_2 - S_1^2) v^i + 6S_1 (v^i)^2 - 6(v^i)^3, \quad (2.23)$$

$i = 1, \dots, n$, where $S_k = S_k(v) = \sum_{i=1}^n (v^i)^k$.

2.4 Solutions with constant h^i

In this paper we deal with the following solutions to equations (2.19) and (2.20)

$$h^i(\tau) = v^i, \quad (2.24)$$

with constant v^i , which correspond to the solutions

$$\beta^i = v^i \tau + \beta_0^i, \quad (2.25)$$

where β_0^i are constants, $i = 1, \dots, n$.

In this case we obtain the metric (2.2) with the exponential dependence of scale factors

$$g = w d\tau \otimes d\tau + \sum_{i=1}^n \varepsilon_i B_i^2 e^{2v^i \tau} dy^i \otimes dy^i, \quad (2.26)$$

where $w = \pm 1$, $\varepsilon_i = \pm 1$ and $B_i > 0$ are arbitrary constants.

For the fixed point $v = (v^i)$ we have the set of polynomial equations

$$G_{ij} v^i v^j - \alpha G_{ijkl} v^i v^j v^k v^l = 0, \quad (2.27)$$

$$\left[2G_{ij} v^j - \frac{4}{3}\alpha G_{ijkl} v^j v^k v^l \right] \sum_{i=1}^n v^i - \frac{2}{3}G_{ij} v^i v^j = 0, \quad (2.28)$$

$i = 1, \dots, n$, where $\alpha = \alpha_2(-w)/\alpha_1$. For $n > 3$ this is a set of forth-order polynomial equations.

The trivial solution $v = (v^i) = (0, \dots, 0)$ corresponds to a flat metric g .

For any non-trivial solution v we have $\sum_{i=1}^n v^i \neq 0$ (otherwise one gets from (2.28) $G_{ij} v^i v^j = \sum_{i=1}^n (v^i)^2 - (\sum_{i=1}^n v^i)^2 = 0$ and hence $v = (0, \dots, 0)$).

The set of equations (2.27) and (2.28) has an isotropic solution $v^1 = \dots = v^n = H$, where

$$n(n-1)H^2 + \alpha n(n-1)(n-2)(n-3)H^4 = 0. \quad (2.29)$$

For $n = 1$: H is arbitrary and for $n = 2, 3$: $H = 0$.

When $n > 3$, the non-zero solution to eq. (2.29) exists only if $\alpha < 0$ and in this case [29, 30]

$$H = \pm \frac{1}{\sqrt{|\alpha|(n-2)(n-3)}}. \quad (2.30)$$

In cosmological case ($w = -1$) this solution takes place when $\alpha_2/\alpha_1 = \alpha < 0$.

The isotropic solution for $n > 3$ gives rise to a very large value of $\dot{G}/G = (n-3)H$, which is forbidden by observational restrictions.

It was shown in [29, 30] that there are no more than three different numbers among v^1, \dots, v^n .²

3 Examples of cosmological solutions obeying the restriction on the variation of G

In this section we consider some solutions to the set of equations (2.27), (2.28) of the following form $v = (H, \dots, H, h, \dots, h)$, where H the “Hubble-like” parameter corresponding to m -dimensional isotropic subspace with $m > 3$ and h is the “Hubble-like” parameter corresponding to l -dimensional isotropic subspace, $l > 2$.

These solutions should satisfy the following conditions:

1. mandatory:

- (a) H and h are real numbers,
- (b) $H > 0$, $h < 0$;

2. desirable:

- (a) $Int = (m-3)H + lh < 0$;
- (b) $-0,65 \cdot 10^{-3} < \frac{\dot{G}}{GH} = -((m-3) + \frac{lh}{H}) < 1,12 \cdot 10^{-3}$.

The first inequality $H > 0$ in the mandatory condition is necessary for a description of accelerated expansion of 3-dimensional subspace, which may describe our Universe, while the second inequality $h < 0$ excludes an enormous (of Hubble’s order) variation \dot{G}/G for $h \geq 0$ and $m > 3$.

The first desirable condition means that the volume scale factor of the internal space $V(\tau) = B \exp(((m-3)H + lh)\tau)$, where $B > 0$ is constant, decreases over time. This condition is a sort of weak extension of a possible restriction for $m = 3$ coming from the unobservability of the “internal space” for all $\tau > \tau_0$. It is also desirable since the negative value of the parameter Int is more probable due to more probable positive value of $\dot{G}/G = -Int$, see (1.9).

²At the moment we were unable to find solutions with three different real “Hubble-like” parameters.

The second desirable condition may be also rewritten by using the parameter $Var = |\frac{\dot{G}}{GH}| = |(m-3) + \frac{lh}{H}|$:

$$Var < 1, 12 \cdot 10^{-3}, \quad \text{for } Int \leq 0; \quad (3.1)$$

$$Var < 0, 65 \cdot 10^{-3}, \quad \text{for } Int \geq 0. \quad (3.2)$$

Here we consider the simplest case, when internal spaces (apart the expansion factors) are flat. The consideration of curved internal spaces will drastically change the equations of motion and may break the existence of solutions with exponential dependence of scale factors. Anyway the inclusion into consideration of curved internal spaces may be worth but need a special treatment, which may be done in a separate work.

3.1 The dependence of “Hubble-like” parameters on m and l

The total dimension of the considered space is $D = n + 1 = (m + l) + 1$, where we have m dimensions expanding with the Hubble parameter $H > 0$ and l dimensions contracting with the “Hubble-like” parameter $h < 0$.

According to this, we rewrite the set of polynomial equations (2.27), (2.28), using the formulas (2.22), (2.23), as follows:

$$\begin{aligned} & H^2(m - m^2) + h^2(l - l^2) - 2mlHh \\ & -\alpha(H^4m(m-1)(m-2)(m-3) + h^4l(l-1)(l-2)(l-3) \\ & + 4H^3hm(m-1)(m-2)l + 4h^3Hl(l-1)(l-2)m \\ & + 6H^2h^2m(m-1)l(l-1)) = 0, \end{aligned} \quad (3.3)$$

$$\begin{aligned} & m(1-m)H^2 - (1/2)lh^2(1+2l) + 2lHh((3/4) - m) \\ & -\alpha(H^4m(m-1)(m-2)(m-3) + H^3hl(m-1)(m-2)(4m-3) \\ & + 3H^2h^2l(m-1)(2lm-2l-m) \\ & + Hh^3l(l-1)(4lm-3l-2m) + h^4l^2(l-1)(l-2)) = 0, \end{aligned} \quad (3.4)$$

$$\begin{aligned} & l(1-l)h^2 - (1/2)mH^2(1+2m) + 2mHh((3/4) - l) \\ & -\alpha(h^4l(l-1)(l-2)(l-3) + h^3Hm(l-1)(l-2)(4l-3) \\ & + 3h^2H^2m(l-1)(2lm-2m-l) \\ & + hH^3m(m-1)(4lm-3m-2l) + H^4m^2(m-1)(m-2)) = 0. \end{aligned} \quad (3.5)$$

Here we put for simplicity $\alpha = \pm 1$ but keep in mind that general α -dependent solution has the following form

$$H(\alpha) = H|\alpha|^{-1/2}, \quad h(\alpha) = h|\alpha|^{-1/2}. \quad (3.6)$$

Due to these relations the parameter $\dot{G}/(GH)$ does not depend upon $|\alpha|$ and hence our simplification is a reasonable one. For any solution (H, h) with $\alpha = \pm 1$ we can find a

proper α which will be in agreement with the present value of the Hubble parameter H_0 (see (1.10))

$$H|\alpha|^{-1/2} = H_0. \quad (3.7)$$

Our numerical analysis (based on Maplesoft Maple) show us that (generically) there are 11 solutions of these equations (for $m > 3$ and $l \geq 3$).

I) The first to mention is, obviously, the zero solution $H_1 = h_1 = 0$.

II) Two other solutions are isotropic ones:

1. if $\alpha = 1$, then $H = h = \pm \sqrt{\frac{1}{(l^2+2lm+m^2-5l-5m+6)}} \cdot i$. We are led to pure imaginary isotropic solutions obeying (2.29). For example, when $m = 9$ and $l = 6$ we receive

$$(a) \quad H_2 = h_2 = \frac{1}{2} \sqrt{\frac{1}{39}} \cdot i;$$

$$(b) \quad H_3 = h_3 = -\frac{1}{2} \sqrt{\frac{1}{39}} \cdot i;$$

2. if $\alpha = -1$, then $H = h = \pm \sqrt{\frac{1}{(l^2+2lm+m^2-5l-5m+6)}} \cdot i$. We are led to isotropic solutions (2.30). When $m = 9$ and $l = 6$ the solutions are:

$$(a) \quad H_2 = h_2 = \frac{1}{2} \sqrt{\frac{1}{39}};$$

$$(b) \quad H_3 = h_3 = -\frac{1}{2} \sqrt{\frac{1}{39}}.$$

III) For $\alpha = \pm 1$ the remaining eight solutions are roots of the following two equations which are given by Maple:

$$\begin{aligned} P(H; m, l) = & 64(m-2)(m-1)^2(m-2+l)(l^2m+lm^2-2l^2 \\ & +2lm-2m^2)(-3+l+m)^2 \cdot H^8 \\ & \mp 128(m-1)^2(-3+l+m)(l^2m(l+m)^2-2l(l+m)^3 \\ & +(l+m)(8l^2+lm+2m^2)-10l^2+4lm-4m^2) \cdot H^6 \\ & +16(m-1)(5l^5m+10l^4m^2+5l^3m^3-6l^5-38l^4m \\ & -49l^3m^2-17l^2m^3+32l^4 \\ & +111l^3m+75l^2m^2+14lm^3-70l^3-130l^2m-14lm^2 \\ & -8m^3+68l^2+4lm+8m^2) \cdot H^4 \\ & \mp 16l(m-1)(l^4+l^3m-3l^3-5l^2m+5l^2+8lm-7l-2m) \cdot H^2 \\ & +l^5-3l^3-2l^2 = 0, \end{aligned} \quad (3.8)$$

$$P(h; l, m) = 0. \quad (3.9)$$

The second equation is obtained from the first one just by swapping parameters m and l and replacing H by h . The solutions to eqs. (3.8), (3.9) should be substituted into eqs. (3.3), (3.4), (3.5), in order to find the solutions (H, h) under consideration.

The closed-form expression for the solution in general case (for any m and l) seems to be very bulky. So, we use Maplesoft Maple to find solutions for certain m and l and test some general features of these solutions⁴:

1. for $\alpha = 1$ in common case we have two pairs of real and quad of complex solutions which differ in signs. (See footnote 4.) For $m = 9$ and $l = 6$ we receive:

- (a) $H_4 \approx -0.2597$ and $h_4 \approx 0.2826$;
- (b) $H_5 \approx 0.2597$ and $h_5 \approx -0.2826$;
- (c) $H_6 \approx -0.1610$ and $h_6 \approx 0.4004$;
- (d) $H_7 \approx 0.1610$ and $h_7 \approx -0.4004$;
- (e) $H_8 \approx -0.0913 - 0.0464 \cdot i$ and $h_8 \approx 0.1456 - 0.0449 \cdot i$;
- (f) $H_9 \approx -0.0913 + 0.0464 \cdot i$ and $h_9 \approx 0.1456 + 0.0449 \cdot i$;
- (g) $H_{10} \approx 0.0913 - 0.0464 \cdot i$ and $h_{10} \approx -0.1456 - 0.0449 \cdot i$;
- (h) $H_{11} \approx 0.0913 + 0.0464 \cdot i$ and $h_{11} \approx -0.1456 + 0.0449 \cdot i$;

2. for $\alpha = -1$ ($m > 3$ and $l \geq 3$) there are no (extra) real solutions. (See footnote 4.)

In the case of $m = 9$ and $l = 6$ we obtain:

- (a) $H_4 \approx -0.2597 \cdot i$ and $h_4 \approx 0.2826 \cdot i$;
- (b) $H_5 \approx 0.2597 \cdot i$ and $h_5 \approx -0.2826 \cdot i$;
- (c) $H_6 \approx 0.1610 \cdot i$ and $h_6 \approx -0.4004 \cdot i$;
- (d) $H_7 \approx -0.1610 \cdot i$ and $h_7 \approx 0.4004 \cdot i$;
- (e) $H_8 \approx -0.0464 - 0.0913 \cdot i$ and $h_8 \approx -0.0449 + 0.1456 \cdot i$;
- (f) $H_9 \approx -0.0464 + 0.0913 \cdot i$ and $h_9 \approx -0.0449 - 0.1456 \cdot i$;
- (g) $H_{10} \approx 0.0464 - 0.0913 \cdot i$ and $h_{10} \approx 0.0449 + 0.1456 \cdot i$;
- (h) $H_{11} \approx 0.0464 + 0.0913 \cdot i$ and $h_{11} \approx 0.0449 - 0.1456 \cdot i$.

It can be seen that none of the solutions for $\alpha = -1$ satisfies our mandatory conditions written in the beginning of this section.

As for $\alpha = 1$, the solutions III.1.b and III.1.d are real and $H > 0$, $h < 0$. It can be verified that in these cases $Int = (m - 3)H + lh < 0$.

Now we have to calculate the variation of the gravitational constant. For $m = 9$ and $l = 6$ we get:

$$Var_5 = \left| \frac{\dot{G}}{GH} \right|_5 = \left| (m - 3) + \frac{l h_5}{H_5} \right| \approx 0.535826;$$

$$Var_7 = \left| \frac{\dot{G}}{GH} \right|_7 = \left| (m - 3) + \frac{l h_7}{H_7} \right| \approx 8.914741.$$

⁴Here we are led to some common features of the solutions just by numerical calculations for a restricted range of numbers m and l .

The first variation is lower: $Var_5 < Var_7$, for $m = 9$ and $l = 6$. This inequality seems to take place for any $m > 3$ and $l \geq 3$. At the moment a rigorous proof of this fact is absent while certain numerical calculations support it. Anyway, here we will focus on the solution (H_5, h_5) , which we consider as a more interesting (for our applications) then (H_7, h_7) . Further we will write H and h instead of H_5 and h_5 in common case.

We can plot behaviour of parameters H , h , Int and Var , for example, keeping fixed $m = 8$ and $m = 10$ and raising l from 5 to 100 by 5. See Figure 1 and Figure 2.

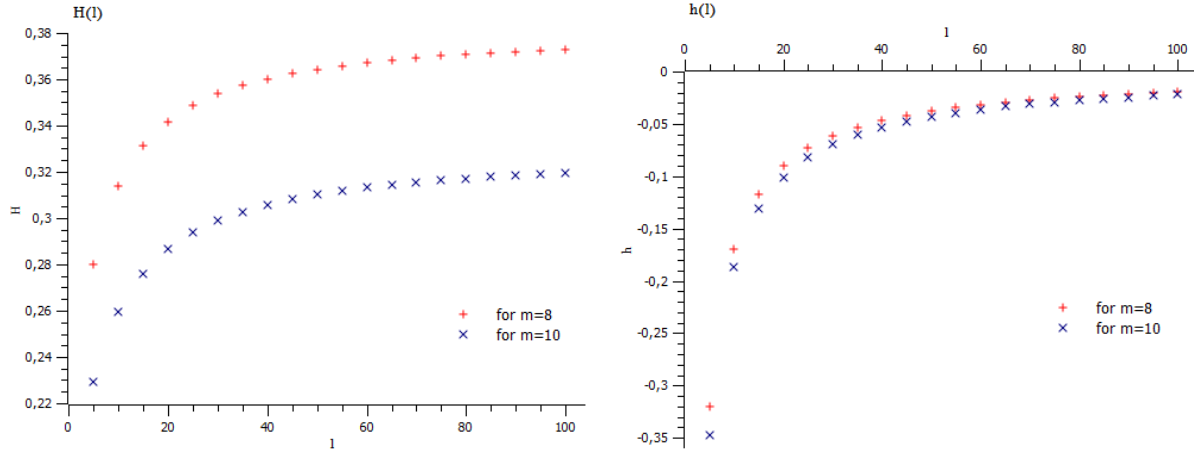


Figure 1: Behavior of “Hubble-like” parameters H and h for fixed $m = 8$ and $m = 10$ while l is changing.

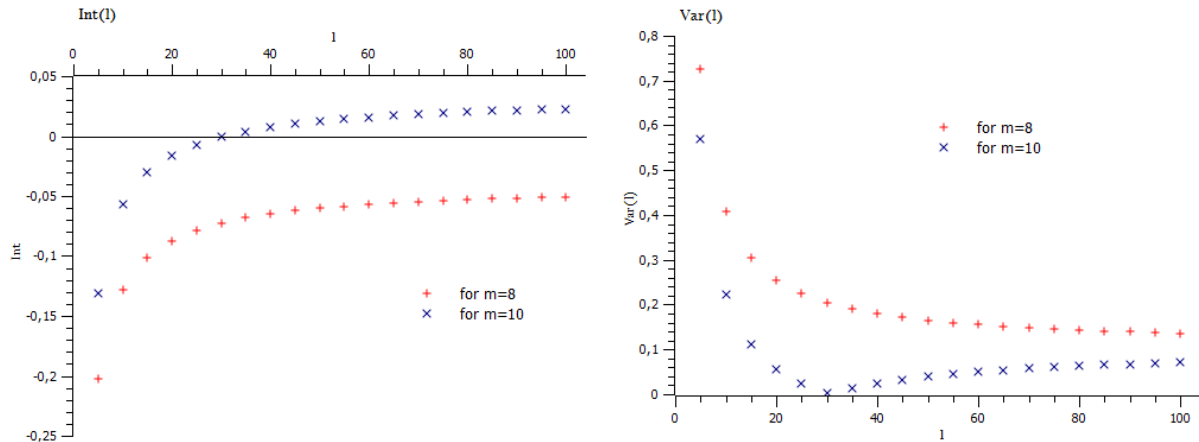


Figure 2: Behavior of the internal space parameter Int and the variation of G parameter Var for fixed $m = 8$ and $m = 10$ while l is changing.

3.2 The limiting values of H , hl , Int and Var for fixed $m \leq 9$

When $m \leq 9$ the internal space parameter Int remains negative, that means that the first desirable condition is satisfied for any l . The variation of G parameter is monotonically decreasing with the increase of l . Moreover, we get finite limits for H and hl as $l \rightarrow +\infty$. In this subsection we obtain these and other limits (for Int and Var) for fixed $m \leq 9$.

Now let us rewrite (3.8) and (3.9) for H and hl keeping only the terms with higher degrees of l :

$$64(m-1)^2(m-2)^2l^5 \cdot H^8 - 128(m-2)(m-1)^2l^5 \cdot H^6 + 16(m-1)(5m-6)l^5 \cdot H^4 - 16(m-1)l^5 \cdot H^2 + l^5 = 0, \quad (3.10)$$

$$\begin{aligned} 64(m-1)(l \cdot h)^8 - 128(m^2 - 2m + 2)(l \cdot h)^6 \\ + (80m^3 - 272m^2 + 224m - 128)(l \cdot h)^4 \\ - 16m(m^3 - 5m^2 + 8m - 2)(l \cdot h)^2 \\ + m^5 - 3m^3 - 2m^2 = 0. \end{aligned} \quad (3.11)$$

Solving these equations we find the limiting values:

$$\lim_{l \rightarrow \infty} H = \frac{1}{2} \sqrt{\frac{2m-2 + \sqrt{2m^2-2m}}{m^2-3m+2}}, \quad (3.12)$$

$$\begin{aligned} \lim_{l \rightarrow \infty} h \cdot l = -\frac{1}{2} \left(\frac{1}{m-2} (m\sqrt{2m(m-1)} + 2m^2 \right. \\ \left. - 2\sqrt{\frac{2((m+2)\sqrt{m(m-1)} + m^2\sqrt{2})(m-1)^2}{\sqrt{m(m-1)}}} \right. \\ \left. - 4m + 4) \right)^{\frac{1}{2}}, \end{aligned} \quad (3.13)$$

$$\lim_{l \rightarrow \infty} Var = \left| (m-3) + \frac{\lim_{l \rightarrow \infty} h \cdot l}{\lim_{l \rightarrow \infty} H} \right|. \quad (3.14)$$

We fill the Table 1 by calculated values for $3 \leq m \leq 9$:

m	$\lim_{l \rightarrow \infty} H$	$\lim_{l \rightarrow \infty} l \cdot h$	$\lim_{l \rightarrow \infty} Int$	$\lim_{l \rightarrow \infty} Var$
3	0.9659258265	-0.7630807575	-0.7630807575	0.7899993318
4	0.6738873385	-1.093021916	-0.4191345775	0.621965355
5	0.5462858555	-1.352249104	-0.259677393	0.475350754
6	0.4709825726	-1.574449592	-0.161501874	0.342904141
7	0.4199717390	-1.772664074	-0.092777118	0.220912766
8	0.3825276619	-1.953613607	-0.040975297	0.107117214
9	0.3535533906	-2.121320344	0	0

Table 1: The limiting values of H , hl , Int and Var parameters as $l \rightarrow \infty$.

For $3 \leq m \leq 8$ the limiting values of Var -parameter are too large. Since $Var(l)$ exceed the limiting values $Var(\infty)$ for $2 < m < 9$ (see Figure 2 for $m = 8$) the restriction (1.8) on variation of G is not satisfied for $m = 3, \dots, 8$ and we are led to unphysical results. This is why we consider in what follows just the cases $m = 9, 10, 12, \dots$

3.3 Infinite series of solutions for $m = 9$

Now we consider the case $\mathbf{m} = 9$. We get the following relations.

For $\mathbf{l} = 2679$:

$\mathbf{H} \approx 0.3531582111$,
 $\mathbf{h} \approx -0.0007910955039$,
 $\mathbf{Int} \approx -0.000395588$,
 $\mathbf{Var} \approx 0.001120145$.

For $\mathbf{l} = 2681$:

$\mathbf{H} \approx 0.3531585062$,
 $\mathbf{h} \approx -0.0007905059028$,
 $\mathbf{Int} \approx -0.000395288$,
 $\mathbf{Var} \approx 0.001119293$.

For $\mathbf{l} = 2680$:

$\mathbf{H} \approx 0.3531583594$,
 $\mathbf{h} \approx -0.0007908005933$,
 $\mathbf{Int} \approx -0.000395434$,
 $\mathbf{Var} \approx 0.001119706$.

For $\mathbf{l} = 2682$:

$\mathbf{H} \approx 0.3531586532$,
 $\mathbf{h} \approx -0.0007902114318$,
 $\mathbf{Int} \approx -0.000395141$,
 $\mathbf{Var} \approx 0.001118876$.

We do not present here exact analytical forms of these solutions in radicals which are bulky ones. For example, the relation for the parameter H , when $l = 2680$, contains (17 times) the radical $\sqrt{32839778319264444823234828568184005}$.

The numerical calculations for fixed $m = 9$ gives us an evidence of monotonically decreasing behaviour of the function $Var(l)$ for $l \geq 2680$ ⁵ as well as the asymptotical relation: $Var(l) \sim A/l$, as $l \rightarrow +\infty$, where $A > 0$. See Figure 3.

Thus, for $m = 9$ there is an infinite series of admissible cosmological solutions with $l = 2680, 2681, \dots$, which satisfy all the conditions imposed. Any of such solution describes accelerated expansion of the three-dimensional factor space with sufficiently small value of the variation of the effective gravitational constant G . This variation may be arbitrary small for a big enough value of l .

⁵A rigorous analytical proof of this fact may be a subject of a separate work.

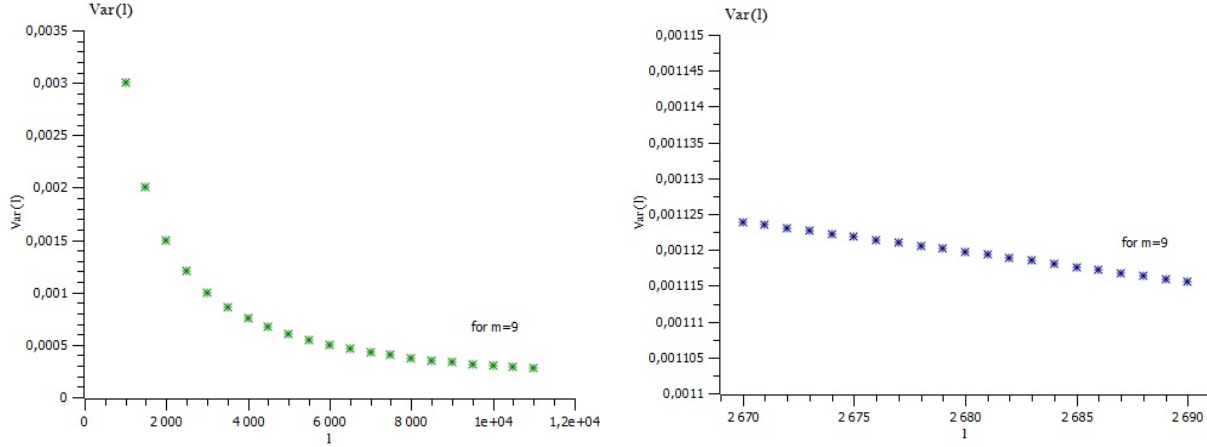


Figure 3: Behavior of the variation of G parameter Var for fixed $m = 9$ while l is changing from 1000 to 11000 by 500 (left) and from 2670 to 2690 by 1 (right).

The infinite series of solutions for $m = 9$ and $l = 2680, 2681, \dots$ starts from the (special) total dimension $D = 2690$. For $D < 2690$ and $m = 9$ the solutions do not obey restriction (1.8) on variation of G and hence are not of interest for our consideration.

3.4 Some solutions for $m > 9$ with minimal Var -parameter

When $m > 9$ the internal space parameter Int becomes positive. As l can only be a natural number we should look for a value of l which gives the minimal magnitude of the variation of G parameter Var . Below we present the calculated values of our parameters ($H, h, Int = (m - 3) \cdot H + l \cdot h$ and $Var = |(m - 3) + \frac{l \cdot h}{H}|$) for each of considered cases:

1. for $m = 10$:

the variation of G parameter is minimal for $l = 31$, see Figure 4,

$$H \approx 0.2996055415,$$

$$h \approx -0.06764217686,$$

$$Int \approx 0.000331307,$$

$$Var \approx 0.001105812.$$

This case is not of particular interest. The radical forms of the solutions are too bulky, so we approximated them. Nevertheless the variation of G parameter is out of the allowed domain and the “internal space” parameter is positive.

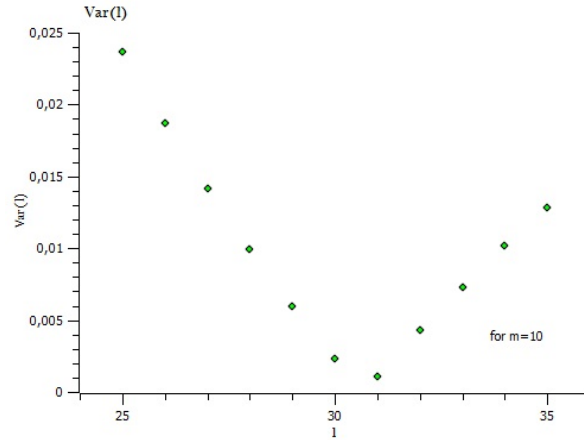


Figure 4: The variation of G parameter for $m = 10$.

2. for $\mathbf{m} = \mathbf{11}$:

the variation of G and the “internal space” parameters are zero for $\mathbf{l} = \mathbf{16}$, see Figure 5,

$$\mathbf{H} = \sqrt{\frac{1}{15}},$$

$$\mathbf{h} = -\frac{1}{2}\sqrt{\frac{1}{15}},$$

$$\mathbf{Int} = 0,$$

$$\mathbf{Var} = 0.$$

This case is the first one with zero variation of G . Also, the exact values of “Hubble-like” parameters ($H = -2h$) in contrast to the previous case have rather simple and compact form.

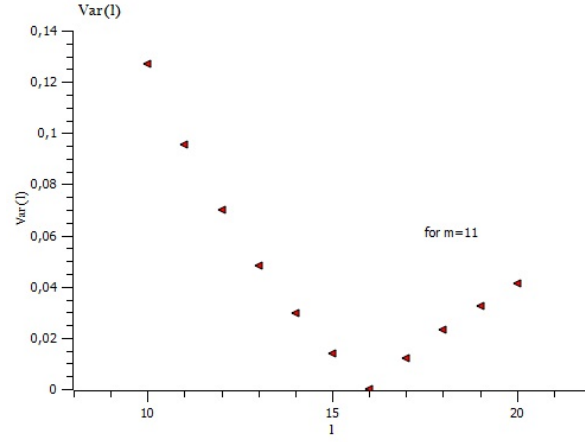


Figure 5: The variation of G parameter for $m = 11$.

3. for $\mathbf{m} = \mathbf{12}$:

the variation of G parameter is minimal for $\mathbf{l} = \mathbf{11}$, see Figure 6,

$$\mathbf{H} \approx 0.2264080186,$$

$$\mathbf{h} \approx -0.1852491999,$$

$$\mathbf{Int} \approx -0.000069032,$$

$$\mathbf{Var} \approx 0.000304899.$$

Here all our four conditions are satisfied. The variation of G parameter is non-zero and the volume of the internal space is decreasing.

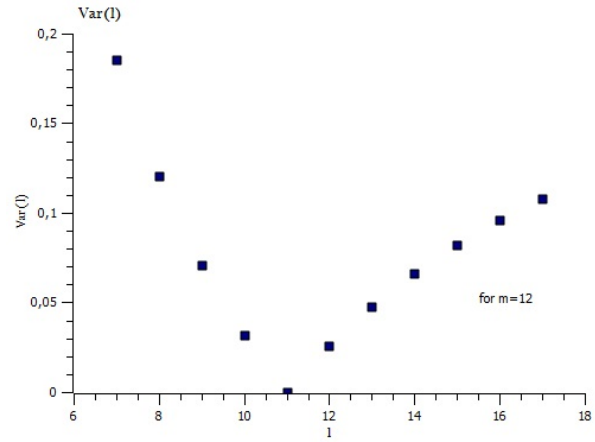


Figure 6: The variation of G parameter for $m = 12$.

4. for $m = 13$:

the variation of G parameter is minimal for $l = 9$ and the “internal space” parameter is positive, see Figure 7,

$$H \approx 0.2039802, h \approx -0.2261006,$$

$$\text{Int} \approx 0.0048967, \text{Var} \approx 0.0240058,$$

for $l = 8$ the variation of G parameter is slightly higher, but the “internal space” parameter is negative, see Figure 8,

$$H \approx 0.1942063, h \approx -0.2498379,$$

$$\text{Int} \approx -0.0052611, \text{Var} \approx 0.0263919.$$

Both cases are excluded by G -dot restrictions.

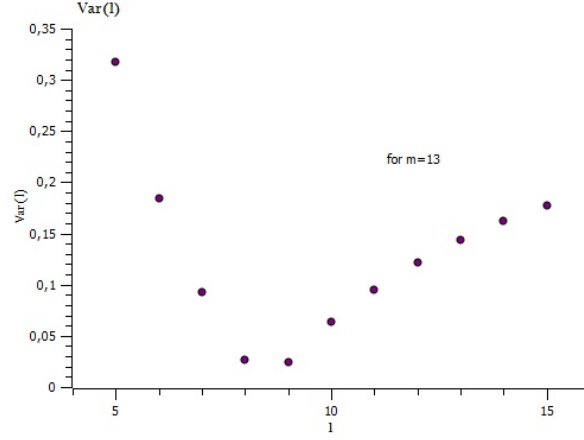


Figure 7: The variation of G parameter for $m = 13$.

5. for $m = 14$:

the variation of G parameter is minimal for $l = 7$, see Figure 8,

$$H \approx 0.1822582965,$$

$$h \approx -0.2863787788,$$

$$\text{Int} \approx 0.000189810,$$

$$\text{Var} \approx 0.00104143.$$

The variation of G parameter exceeds our limits, and the condition of the volume contraction of the “inner space” is not met.

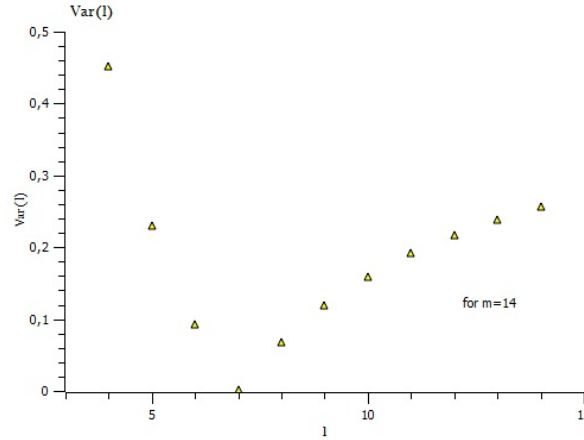


Figure 8: The variation of G parameter for $m = 14$.

6. for $\mathbf{m} = 15$:

the variation of G and the “internal space” parameters are zero for $\mathbf{l} = 6$, see Figure 9,

$$\mathbf{H} = \frac{1}{6},$$

$$\mathbf{h} = -\frac{1}{3},$$

$$\mathbf{Int} = 0,$$

$$\mathbf{Var} = 0.$$

This is the second case with zero variation of G . The exact values of “Hubble-like” parameters ($H = -\frac{1}{2}h$) have simple and compact form.

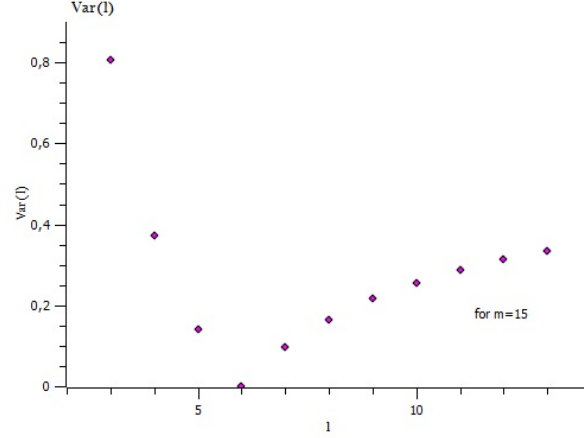


Figure 9: The variation of G parameter for $m = 15$.

Now we will reverse our method and look for the solutions with minimal variation of G for fixed l instead of m . The calculations lead to the following rule: the lesser is l the greater is an appropriate m which gives the minimum of Var . As we consider $l \geq 3$ and for $l = 6$ the solution with minimal variation of G parameter is already found, we should examine only three cases:

1. for $\mathbf{l} = 5$:

the variation of G parameter is minimal for $\mathbf{m} = 17$, see Figure 10,

$$\mathbf{H} \approx 0.1447364880,$$

$$\mathbf{h} \approx -0.4041874693,$$

$$\mathbf{Int} \approx 0.005373486,$$

$$\mathbf{Var} \approx 0.017555134.$$

None of the desirable conditions are satisfied.

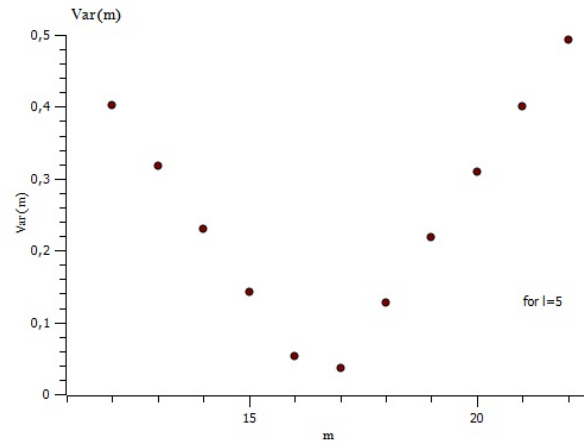


Figure 10: The variation of G parameter for $l = 5$.

2. for $l = 4$:

- the variation of G parameter is minimal for $m = 20$, see Figure 11,

- $H \approx 0.1220672556$

- $h \approx -0.5176845111$

- $\text{Int} \approx 0.004405301$

- $\text{Var} \approx 0.03608913$

The amount of variation is too high and the “internal space” parameter is positive.

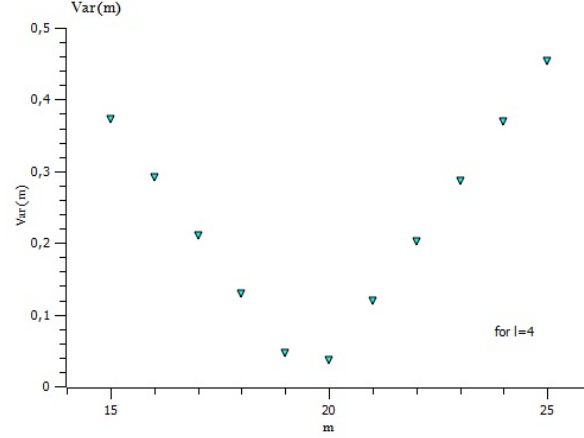


Figure 11: Variation of G for $l = 4$.

3. for $l = 3$:

the variation of G parameter is minimal for $m = 28$, see Figure 12,

$H \approx 0.09202826388$,

$h \approx -0.7765606872$,

$\text{Int} \approx 0.000272078$,

$\text{Var} \approx 0.00295646$.

The “internal space” parameter is also positive and the variation of G parameter exceeds the limits imposed.

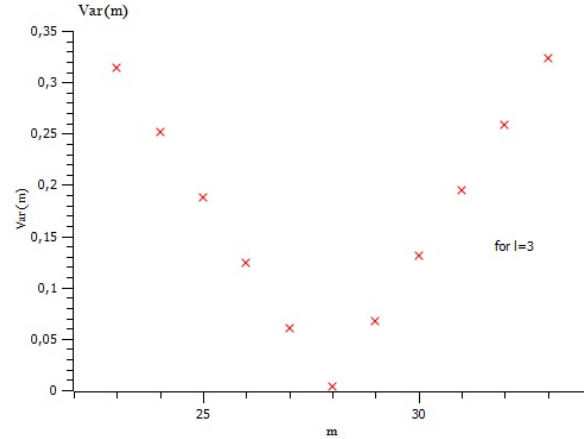


Figure 12: The variation of G parameter for $l = 3$.

Thus, in this subsection we have obtained cosmological solutions for $m > 9$, which satisfy all four conditions for the following cases:

1. $m = 11, l = 16$ (zero variation of G);
2. $m = 12, l = 11$;
3. $m = 15, l = 6$ (zero variation of G).

It should be noted that for $m = 3$ and $l = 2$ the solution with $H \approx 0,750173$ and $h \approx -0,541715$ was found earlier in [16]. For this solution we have a contracting “internal space” but the variation of G is a huge one (\dot{G}/G is of Hubble parameter order). Recently, an exact analytic form of this solution was obtained in [17].

4 Conclusions

We have considered the $(n + 1)$ -dimensional Einstein-Gauss-Bonnet (EGB) model. By using the ansatz with diagonal cosmological type metrics, we have found solutions with exponential dependence of scale factors with respect to “synchronous-like” variable τ .

In cosmological case ($w = -1$) these solutions describe an exponential expansion of “our” 3-dimensional factor-space with the Hubble parameter $H > 0$ and obey the observational constraints on the temporal variation of the effective gravitational constant G . Any solution describes $(m - 3 + l)$ -dimensional “internal space”, which is anisotropic: it is expanding in $(m - 3)$ dimensions with the Hubble rate $H > 0$ and contracting in l dimensions.

These solutions were found (in numerical or analytical forms) for the following cases:

1. **$m = 9, l \geq 2680$** (variation of G tends to 0 as $l \rightarrow +\infty$);
2. **$m = 11, l = 16$** (variation of G is zero);
3. **$m = 12, l = 11$** ;
4. **$m = 15, l = 6$** (variation of G is zero).

Thus, we have shown that it is possible in the framework of EGB model to describe the accelerated expansion of the three-dimensional factor space with sufficiently small (or even zero) value of the variation of the effective gravitational constant G . For the case $w = 1$ we have obtained by product a family of static configurations which may be of interest within some other possible applications.

Here we have considered a gravitational model in more than 4-dimensions. In such a case the Gauss-Bonnet term gives non-trivial contributions to the generalized Einstein field equations. In particular we have shown that there are cosmological solutions in agreement with observations when “projected on the $(3 + 1)$ -dimensional physical space-time. For the sake of simplicity, we restrict ourselves to vacuum solutions in multidimensional gravity with the Gauss-Bonnet term. Such ansatz may be considered as a part of a general “geometrical program aimed at the explanation of dark energy in 4-dimensional space, e.g. by using extra dimensions and modified equation of motions just without matter sources. This is a first step. The inclusion of matter sources (e.g. anisotropic fluid) will be the next step, e.g. as a subject of a next publication.

References

- [1] B. Zwiebach, Curvature squared terms and string theories, *Phys. Lett. B* **156**, 315 (1985).
- [2] D. Gross and E. Witten, Superstrings modifications of Einstein's equations, *Nucl. Phys. B* **277**, 1 (1986).
- [3] D.J. Gross and J. H. Sloan, The quartic effective action for the heterotic string, *Nucl. Phys. B* **291**, 41 (1987).
- [4] R.R. Metsaev and A.A. Tseytlin, Two loop beta function for the generalized bosonic sigma model, *Phys. Lett. B* **191**, 354 (1987).
- [5] R.R. Metsaev and A.A. Tseytlin, Order alpha-prime (two loop) equivalence of the string equations of motion and the sigma model Weyl invariance conditions: dependence on the dilaton and the antisymmetric tensor, *Nucl. Phys. B* **293**, 385 (1987).
- [6] S. Nojiri and S.D. Odintsov, Introduction to modified gravity and gravitational alternative for Dark Energy, *Int. J. Geom. Meth. Mod. Phys.* **4**, 115-146 (2007); hep-th/0601213.
- [7] G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov and S. Zerbini, One-loop effective action for non-local modified Gauss-Bonnet gravity in de Sitter space, arXiv: 0905.0543.
- [8] H. Ishihara, Cosmological solutions of the extended Einstein gravity with the Gauss-Bonnet term, *Phys. Lett. B* **179**, 217 (1986).
- [9] N. Deruelle, On the approach to the cosmological singularity in quadratic theories of gravity: the Kasner regimes, *Nucl. Phys. B* **327**, 253-266 (1989).
- [10] E. Elizalde, A.N. Makarenko, V.V. Obukhov, K.E. Osetrin and A.E. Filippov, Stationary vs. singular points in an accelerating FRW cosmology derived from six-dimensional Einstein-Gauss-Bonnet gravity, *Phys. Lett. B* **644**, 1-6 (2007); hep-th/0611213.
- [11] K. Bamba, Z.-K. Guo and N. Ohta, Accelerating Cosmologies in the Einstein-Gauss-Bonnet theory with dilaton, *Prog. Theor. Phys.* **118**, 879-892 (2007); arXiv: 0707.4334.
- [12] A. Toporensky and P. Tretyakov, Power-law anisotropic cosmological solution in 5+1 dimensional Gauss-Bonnet gravity, *Grav. Cosmol.* **13**, 207-210 (2007); arXiv: 0705.1346.
- [13] I.V. Kirnos, A.N. Makarenko, S.A. Pavluchenko and A.V. Toporensky, The nature of singularity in multidimensional anisotropic Gauss-Bonnet cosmology with a perfect fluid, *General Relativity and Gravitation* **42**, 2633-2641 (2010); arXiv: 0906.0140.
- [14] S.A. Pavluchenko and A.V. Toporensky, A note on differences between (4+1)- and (5+1)-dimensional anisotropic cosmology in the presence of the Gauss-Bonnet term, *Mod. Phys. Lett. A* **24**, 513-521 (2009).
- [15] I.V. Kirnos and A.N. Makarenko, Accelerating cosmologies in Lovelock gravity with dilaton, *Open Astronomy Journal* **3**, 37-48 (2010); arXiv: 0903.0083.

- [16] D.A. Ratanov, Multidimensional cosmology in the model of gravitation with Gauss-Bonnet adding. Magister dissertation (under the supervision of V.D. Ivashchuk). Inst. of Gravit. and Cosmol., PFUR, Moscow, 2011, p. 1-30. (In Russian)
- [17] D. Chirkov, S. Pavluchenko and A. Toporensky, Exact exponential solutions in Einstein-Gauss-Bonnet flat anisotropic cosmology, *Mod. Phys. Lett. A* **29**, 1450093 (11pp.) (2014); arXiv:1401.2962
- [18] A.G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant, *The Astronomical Journal.*, **116**, 1009-1038 (1998).
- [19] S. Perlmutter et al. Measurements of Omega and Lambda from 42 High-Redshift Supernovae. *The Astrophysical Journal*, **517**, 565-586 (1999).
- [20] M. Kowalski, D. Rubin et al., Improved cosmological constraints from new, old and combined supernova datasets; arXiv: 0804.4142.
- [21] K.A. Bronnikov, V.D. Ivashchuk and V.N. Melnikov, Time Variation of Gravitational Constant in Multidimensional Cosmology, *Nuovo Cimento* **B 102**, P. 209-215 (1998).
- [22] J.-M. Alimi, V.D. Ivashchuk, S.A. Kononogov and V.N. Melnikov, Multidimensional cosmology with anisotropic fluid: acceleration and variation of G, *Grav. Cosmol.* **12**, No. 2-3, 173-178 (2006).
- [23] V.D. Ivashchuk, S.A. Kononogov, V.N. Melnikov and M. Novello, Non-singular solutions in multidimensional cosmology with perfect fluid: acceleration and variation of G, *Grav. Cosmol.* **12**, No. 4, 273-278 (2006); hep-th/0610167.
- [24] V.N. Melnikov, Models of G time variations in diverse dimensions, *Front. Phys. China.* **4**, 75-93 (2009).
- [25] V.D. Ivashchuk and V.N. Melnikov, On Time Variations of Gravitational and Yang-Mills Constants in a Cosmological Model of Superstring Origin, *Grav. Cosmol.* **20**, No. 1, 26-29 (2014); arxiv: 1401.5491.
- [26] E.V. Pitjeva, Updated IAA RAS Planetary Ephemerides-EPM2011 and Their Use in Scientific Research, *Astron. Vestnik* **47**, No 5, 419-435 (2013), arXiv: 1308.6416.
- [27] P.A.R. Ade et al. [Planck Collaboration], Planck 2013 results. I. Overview of products and scientific results, *Astronomy and Astrophysics* **571**, A1 (2014); arXiv: 1303.5076.
- [28] S.A. Pavluchenko, On the general features of Bianchi-I cosmological models in Lovelock gravity, *Phys. Rev. D* **80**, 107501 (2009); arXiv: 0906.0141.
- [29] V.D. Ivashchuk, On anisotropic Gauss-Bonnet cosmologies in $(n + 1)$ dimensions, governed by an n -dimensional Finslerian 4-metric, *Grav. Cosmol.* **16**, No 2, 118-125 (2010); arXiv: 0909.5462.
- [30] V.D. Ivashchuk, On cosmological-type solutions in multidimensional model with Gauss-Bonnet term, *International Journal of Geometric Methods in Modern Physics* **7**, No. 5, 797-819 (2010); arXiv: 0910.3426.